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On "The Game of $1+n$ Cars"

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For $n=1$ the necessary and sufficient condition for the existence of the evasion strategy, at any initial situation, has been formulated, see [1]. Here we show when there exists the evasion strategy for all $n \in \mathbb{N}$. The method used here may be applied in some other examples.

1. GENERAL CONSTRUCTION

Denote by U the set of all measurable functions u: $[0, \infty) \to [-\bar{u}, \bar{u}]$ and by W the set of all measurable functions w: $[0, \infty) \to [-\bar{w}, \bar{w}]$.

For $t \in [0, \infty)$, $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ and for $u \in U$ denote by $x = (x_1, x_2, x_3) = (x_1[t, a, u], x_2[t, a, u], x_3[t, a, u]) = x[t, a, u]$ the solution of the differential equation

$$\begin{aligned} x_1'(s) &= v_E \cos x_3(s), & x_1(t) &= a_1, \\ x_2'(s) &= v_E \sin x_3(s), & x_2(t) &= a_2, \\ x_3'(s) &= u(s), & x_3(t) &= a_3. \end{aligned} \tag{1.1}$$

Similarly, for $t \in [0, \infty)$, $b = (b_1, b_2, b_3) \in \mathbb{R}^3$ and for $w \in W$ denote by $y = (y_1, y_2, y_3) = (y_1[t, b, w], y_2[t, b, w], y_3[t, b, w]) = y[t, b, w]$ the solution of the equation

$$\begin{aligned} y_1'(s) &= v_P \cos y_3(s), & y_1(t) &= b_1, \\ y_2'(s) &= v_P \sin y_3(s), & y_2(t) &= b_2, \\ y_3'(s) &= w(s), & y_3(t) &= b_3. \end{aligned} \tag{1.2}$$

These equations describe the so-called "Game of Two Cars," see [3].

Now, for $t \in [0, \infty)$, $a, b \in \mathbb{R}^3$ we take

$$X(t, a) = x[t, a, U] \qquad \text{and} \qquad Y(t, b) = y[t, b, W].$$

Let $n \geqslant 1$, $t \in [0, \infty)$ and $a, b^i \in \mathbb{R}^3$, $i = 1, 2, \ldots, n$, be fixed.

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DEFINITION 1.1. We say that a function $e: Y(t, b^1) \times \cdots \times Y(t, b^n) \to X(t, a)$ is a strategy of the player E (evader), in the game $(X, Y, a, b^1, ..., b^n, t)$, if

(SE) for any $y \in Y(t, b^1) \times \cdots \times Y(t, b^n)$ there exists a closed, well ordered set $C \in [t, \infty)$ with $\min C = t$, $\sup C = \infty$ and such that for any $c \in C$ and $\tilde{y} \in Y(t, b^1) \times \cdots \times Y(t, b^n)$, if $\tilde{y}|_{[t,c]} = y|_{[t,c]}$, then $e(\tilde{y})|_{[t,c']} = e(y)|_{[t,c']}$, where $c' = \min\{s \in C: c < s\}$.

The set of all such strategies will be denoted by $E(X, Y, a, b^1, ..., b^n, t)$.

Further we will say that the set C from the condition (SE) is determined by e and y. Of course, for fixed e and y there could be many such C's, e.g., for e being constant function.

DEFINITION 1.2. We say that a function $p: X(t, a) \to Y(t, b^1) \times \cdots \times Y(t, b^n)$ is a strategy of the players $P_1, ..., P_n$ (pursuers), in the game $(X, Y, a, b^1, ..., b^n, t)$, if

(SP) for any $s \in [t, \infty)$ and $x, \tilde{x} \in X(t, a)$, if $x|_{[t,s]} = \tilde{x}|_{[t,s]}$, then $p(x)|_{[t,s]} = p(\tilde{x})|_{[t,s]}$.

The set of all such strategies will be denoted by $P(X, Y, a, b^1, ..., b^n, t)$.

We say that $e \in E(X, Y, a, b^1, ..., b^n, t)$ wins, in the game $(X, Y, a, b^1, ..., b^n, t)$, if for any $y = (y^1, ..., y^n) \in Y(t, b^1) \times \cdots \times Y(t, b^n)$ and $x = e(y)$ we have

$$(x_1(s), x_2(s)) \neq (y_1^i(s), y_2^i(s)),$$

for all $s \in [t, \infty)$ and $i = 1, 2, ..., n$. If there exists $e \in E(X, X, a, b^1, ..., b^n, t)$ which wins in the game $(X, Y, a, b^1, ..., b^n, t)$, then we also say that the player E wins in this game.

Conversely, we say that $p \in P(X, Y, a, b^1, ..., b^n, t)$ wins in the game $(X, Y, a, b^1, ..., b^n, t)$, if for any $x \in X(t, a)$ and $y = (y^1, ..., y^n) = p(x)$ there exists $s \in [t, \infty)$ and $i \in \{1, 2, ..., n\}$ for which

$$(x_1(s), x_2(s)) = (y_1^i(s), y_2^i(s)).$$

If there exists $p \in P(X, Y, a, b^1, ..., b^n, t)$ which wins in the game $(X, Y, a, b^1, ..., b^n, t)$, then we also say that the players $P_1, ..., P_n$ win in this game.

Now we are going to formulate some properties of strategies of the evader E and the pursuers $P_1, ..., P_n$.

Let $\pi_i: Y(t, b^1) \times \cdots \times Y(t, b^n) \to Y(t, b^i)$ be defined by the formula

$$\pi_i(y^1, ..., y^n) = y^i,$$

for all $(y^1, ..., y^n) \in Y(t, b^1) \times \cdots \times Y(t, b^n)$ and $i = 1, 2, ..., n$.

PROPOSITION 1.1. *(1) $p \in P(X, Y, a, b^1, ..., b^n, t)$ iff $\pi_i p \in P(X, Y, a, b^i, t)$, for every $i = 1, 2, ..., n$.*

(2) If $e \in E(X, Y, a, b^1, ..., b^n, t)$, then for any, $y = (y^1, ..., y^n) \in Y(t, b^1) \times \cdots \times Y(t, b^n)$ and every $i = 1, 2, ..., n$ we have $e(y^1, ..., y^{i-1}, \cdot, y^{i+1}, ..., y^n) \in E(X, Y, a, b^i, t)$.

The proof of (1) is easy. The proof of (2) is given in [6, Theorem 1.3].

PROPOSITION 1.2. *For any $e \in E(X, Y, a, b^1, ..., b^n, t)$ and any $p \in P(X, Y, a, b^1, ..., b^n, t)$ there exists the unique pair $(x, y) \in X(t, a) \times (Y(t, b^1) \times \cdots \times Y(t, b^n))$ such that*

$$x = e(y) \qquad \text{and} \qquad y = p(x).$$

The proof of this Theorem is given in [6, Theorem 1.1].

It easily follows from Proposition 1.2 that the situation in which both the evader and the pursuers have simultaneously winning strategies is impossible.

Now we will define a particular case of so-called ε-strategy, see [5].

Suppose that f: $[0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to X([0, \infty) \times \mathbb{R}^3)$ and δ: $[0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to (0, \infty)$ satisfy the following conditions: $f(t, a, b) \in X(t, a)$, for all $(t, a, b) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$ and $\inf\{\delta(t, a, b)$: $(t, a, b) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3\} > 0$.

PROPOSITION 1.3. *Under above assumptions, for any $(t, a, b) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$ one can find the unique strategy $e \in E(X, Y, a, b, t)$ such that for each $y \in Y(t, b)$ there exists the set C determined by e and y for which, if $c \in C$ and $c' = \min\{s \in C: c < s\}$, then*

$$c' = c + \delta(c, (e(y))(c), y(c))$$

and

$$(e(y))(s) = (f(c, (e(y))(c), y(c)))(s), \qquad \text{for} \quad s \in [c, c'].$$

The proof may be easily carried out by induction.

DEFINITION 1.3. We denote the strategy e from Proposition 1.3 by $[f, \delta, a, b, t]$.

DEFINITION 1.4. Assume that $[f, \delta, a, b, t] \in E(X, Y, a, b, t)$ and ρ: $[0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to (0, \infty)$. We say that a strategy $[g, \delta, a, b, t] \in E(X, Y, a, b, t)$ is ρ-closed to $[f, \delta, a, b, t]$, if $\| g(\bar{t}, \bar{a}, \bar{b})(s) - f(\bar{t}, \bar{a}, \bar{b})(s) \| \leqslant \rho(\bar{t}, \bar{a}, \bar{b})$, for all $(\bar{t}, \bar{a}, \bar{b}) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$ and $s \in [\bar{t}, \bar{t} + \delta(\bar{t}, \bar{a}, \bar{b})]$ (where $\| \ \|$ denotes the euclidean norm in the spaces $\mathbb{R}^m$, for $m \in \mathbb{N}$).

DEFINITION 1.5. We say that a strategy $|f, \delta, a, b, t| \in E(X, Y, a, b, t)$ satisfies a condition (K) with the liberty of movement ρ, if all strategies $|g, \delta, a, b, t|$ ρ-closed to $|f, \delta, a, b, t|$ satisfy that condition (K).

EXAMPLE 1.1. Let $Z = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_2 \geqslant 0\}$. We say that strategy $e \in E(X, Y, a, b^1, ..., b^n, t)$ keeps the player E in the set Z, if $(e(y))(s) \in Z$, for all $y \in Y(t, b^1) \times \cdots \times Y(t, b^n)$ and $s \in [t, \infty)$. Assume that $n = 1$, $(t, a, b) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$ and $a_2 > 0$.

If there exists $k \in \mathbb{Z}$ (where $\mathbb{Z}$ denotes the set of all integer numbers) for which $a_3 \in (2k\pi, (2k+1)\pi)$, then take $\rho(\bar{t}, \bar{a}, \bar{b}) = a_2$, for all $(\bar{t}, \bar{a}, \bar{b}) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$.

If $a_3 \notin (2k\pi, (2k+1)\pi)$, for all $k \in \mathbb{Z}$, then assume that $a_2 > (1 - |\cos a_3|) v_E/\bar{u}$ (where v_E and $\bar{u}$ are given in the formula (1.1)) and take

$$\rho(\bar{t}, \bar{a}, \bar{b}) = a_2 - (1 - |\cos a_3|) v_E/\bar{u}, \qquad \text{for all} \quad (\bar{t}, \bar{a}, \bar{b}) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3.$$

Now, for $(\bar{t}, \bar{a}, \bar{b}) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$ we put

$$\delta(\bar{t}, \bar{a}, \bar{b}) = \pi/2\bar{u}, \qquad f(\bar{t}, \bar{a}, \bar{b}) = x|\bar{t}, \bar{a}, \bar{u}|,$$

if there exists $k \in \mathbb{Z}$ such that

$$a_3 \in [-\pi/2 + 2k\pi, \pi/2 + 2k\pi]$$

and

$$f(\bar{t}, \bar{a}, \bar{b}) = x|\bar{t}, \bar{a}, -\bar{u}|, \qquad \text{if there is not such } k.$$

It is not hard to show that the strategy $|f, \delta, a, b, t|$ keeps the player E in the set Z with the liberty of movement ρ.

DEFINITION 1.6. Let $x \in X(t, a)$ and $\varepsilon, T \in (0, \infty)$ be fixed. We say that $e \in E(X, Y, a, b^1, ..., b^n, t)$ wins (in the game $(X, Y, a, b^1, ..., b^n, t)$) in the neighbourhood ε of the trajectory x on the interval $[t, t+T]$, if $((e(y))_1(s), (e(y))_2(s)) \neq (y_1^i(s), y_2^i(s))$ and $\|(e(y))(s) - x(s)\| \leqslant \varepsilon$, for all $y = (y^1, ..., y^n) \in Y(t, b^1) \times \cdots \times Y(t, b^n)$, $s \in [t, t+T]$ and $i = 1, 2, ..., n$.

DEFINITION 1.7. We say that the player E wins in the game $(X, Y)_n$, along each trajectory, if for any $(t, a, b^i) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$ such that $(a_1, a_2) \neq (b_1^i, b_2^i)$, $i = 1, 2, ..., n$, for any $x \in X(t, a)$ and for all $\varepsilon, T \in (0, \infty)$ there exists $e \in E(X, Y, a, b^1, ..., b^n, t)$ which wins in the neighbourhood ε of the trajectory x on the interval $[t, t+T]$.

PROPOSITION 1.4. *Suppose that* $|f, \delta, a, b^1, t| \in E(X, Y, a, b^1, t)$ *satisfies a condition* (K) *with the liberty of movement* ρ *and that the player* E *wins* *along each trajectory in the game* $(X, Y)_2$. *Then, for any* $b^2 \in \mathbb{R}^3$, *such that* $(a_1, a_2) \neq (b_1^2, b_2^2)$, *there exists* $e \in E(X, Y, a, b^1, b^2, t)$ *which satisfies the following conditions*:

$$e(\cdot, y^2) \text{ satisfies the condition } (K) \text{ and}$$

$$e(y^1, \cdot) \text{ wins in the game } (X, Y, a, b^2, t),$$

for all $(y^1, y^2) \in Y(t, b^1) \times Y(t, b^2)$.

Proof. For $\bar{t} \in [0, \infty)$, $\bar{a}, \bar{b}^1, \bar{b}^2 \in \mathbb{R}^3$, such that $(\bar{a}_1, \bar{a}_2) \neq (\bar{b}_1^2, \bar{b}_2^2)$, denote by $G(\bar{t}, \bar{a}, \bar{b}^1, \bar{b}^2)$ a strategy from $E(X, Y, \bar{a}, \bar{b}^2, \bar{t})$ which wins in the game $(X, Y, \bar{a}, \bar{b}^2, \bar{t})$ in the neighbourhood $\rho(\bar{t}, \bar{a}, \bar{b}^1)$ of the trajectory $f(\bar{t}, \bar{a}, \bar{b}^1)$ on the interval $[\bar{t}, \bar{t} + \delta(\bar{t}, \bar{a}, \bar{b}^1)]$.

Let $b^2 \in \mathbb{R}^3$, such that $(a_1, a_2) \neq (b_1^2, b_2^2)$ be fixed. For $y^2 \in Y(t, b^2)$, $\bar{t} \in [t, \infty)$, $\bar{a}, \bar{b} \in \mathbb{R}^3$ take

$$g_{y^2}(\bar{t}, \bar{a}, \bar{b}) = G(\bar{t}, \bar{a}, \bar{b}, y^2(\bar{t}))(y^2|_{[\bar{t}, \infty)}), \qquad \text{if} \quad (\bar{a}_1, \bar{a}_2) \neq (y_1^2(\bar{t}), y_2^2(\bar{t}))$$

and

$$g_{y^2}(\bar{t}, \bar{a}, \bar{b}) = f(\bar{t}, \bar{a}, \bar{b}) \qquad \text{in the other case.}$$

Now, for $y = (y^1, y^2) \in Y(t, b^1) \times Y(t, b^2)$ we put

$$e(y) = [g_{y^2}, \delta, a, b^1, t](y^1).$$

It is not hard to show that $e \in E(X, Y, a, b^1, b^2, t)$ and that e satisfies the needed conditions.

Definition 1.8. We say that the player E wins easy along each trajectory in the game (X, Y), if for any $\varepsilon, T \in (0, \infty)$ there is a function ρ: $[0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to (0, \infty)$ such that for all $(t, a, b) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$ for which $(a_1, a_2) \neq (b_1, b_2)$ and for all $x \in X(t, a)$ there exists a strategy $[f, \delta, a, b, t] \in E(X, Y, a, b, t)$ which wins in the neighbourhood ε of the trajectory x on the interval $[t, t+T]$ with the liberty of movement ρ.

Proposition 1.5. *If the player E wins easy along each trajectory in the game* (X, Y) *and wins along each trajectory in the game* $(X, Y)_n$, *then he wins along each trajectory in the game* $(X, Y)_{1+n}$.

The proof is similar to the proof of Proposition 1.4.

Corollary 1.1. *If the player E wins easy along each trajectory in the game* (X, Y), *then for all* $n \in \mathbb{N}$ *he wins along each trajectory in the game* $(X, Y)_n$.

2. Main Theorem

THEOREM 2.1. *If $v_E > v_P$ and $v_E \bar{u} \geqslant v_P \bar{w}$, then the player E wins easy along each trajectory in the game (X, Y). (The constants v_E, v_P, $\bar{u}$ and $\bar{w}$ are given in the formulas* (1.1) *and* (1.2)).

The proof of this Theorem needs the following lemmas.

LEMMA 2.1. *For all $t \in [0, \infty)$, a, $b \in \mathbb{R}^3$, $x \in X(t, a)$, $y \in Y(t, b)$ and $s \in [t, \infty)$ the inequalites*

$$\|x(s) - a\| \leqslant \sqrt{v_E^2 + \bar{u}^2}\,(s-t),$$
$$\|y(s) - b\| \leqslant \sqrt{v_P^2 + \bar{w}^2}\,(s-t)$$

hold.

Proof. The proof is easy.

LEMMA 2.2. *For all $t \in [0, \infty)$, a, $\tilde{a} \in \mathbb{R}^3$, $u \in U$ and $s \in [t, \infty)$ we have*

$$\|x[t, a, u](s) - x[t, \tilde{a}, u](s)\| \leqslant \|a - \tilde{a}\|(1 + \sqrt{2}\, v_E(s-t) + 2\sqrt{v_E(s-t)}).$$

Proof. It is enough to notice that

$$|x_i[t, a, u](s) - x_i[t, \tilde{a}, u](s)| \leqslant |a_i - \tilde{a}_i| + v_E|a_3 - \tilde{a}_3|(s-t), \qquad \text{for} \quad i = 1, 2$$

and

$$|x_3[t, a, u](s) - x_3[t, \tilde{a}, u](s)| = |a_3 - \tilde{a}_3|,$$
$$\text{for all } t \in [0, \infty), a, \tilde{a} \in \mathbb{R}^3\ u \in U \text{ and } s \in [t, \infty).$$

Denote by $\langle a, b\rangle$ the euclidean scalar product of the vectors $a, b \in \mathbb{R}^2$ and take

$$M = \{(t, a, b) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 : \langle (a_1, a_2) - (b_1, b_2), (\cos a_3, \sin a_3)\rangle \geqslant 0\}.$$

LEMMA 2.3. *There is $H \in (0, \infty)$ and increasing function $\sigma\colon [0, H] \to [0, \infty)$ such that, if $(t, a, b) \in M$, then*

$$\|(x_1(s), x_2(s)) - (y_1(s), y_2(s))\| \geqslant \sigma(s-t),$$

for all $x \in X(t, a)$, $y \in Y(t, b)$ and $s \in [t, t+H]$.

Proof. Take

$$H=\frac{v_E-v_P}{v_E\bar{u}+v_P\bar{w}},\qquad \sigma(\tau)=(v_E-v_P)\,\tau-\frac{v_E\bar{u}+v_P\bar{w}}{2}\tau^2,\quad \text{for } \tau\in[0,H]$$

and suppose that $(t,a,b)\in M$, $x\in X(t,a)$ and $y\in Y(t,b)$. Let $\tilde{x}=(x_1,x_2)$ and $\tilde{y}=(y_1,y_2)$. Then

$$\begin{aligned}
\|\tilde{x}(s)-\tilde{y}(s)\| &\geqslant \left\langle \tilde{x}(s)-\tilde{y}(s),\frac{\tilde{x}'(t)}{\|\tilde{x}'(t)\|}\right\rangle \\
&= \Big\langle \tilde{x}(t)-\tilde{y}(t)+(s-t)(\tilde{x}'(t)-\tilde{y}'(t)) \\
&\quad +\int_t^s\left(\int_t^\tau (\tilde{x}''(\xi)-\tilde{y}''(\xi))\,d\xi\right)d\tau,\frac{\tilde{x}'(t)}{v_E}\Big\rangle \\
&= \left\langle \tilde{x}(t)-\tilde{y}(t),\frac{\tilde{x}'(t)}{v_E}\right\rangle+(\|\tilde{x}'(t)\|^2-\langle\tilde{y}'(t),\tilde{x}'(t)\rangle)\frac{s-t}{v_E} \\
&\quad +\int_t^s\left(\int_t^\tau\left\langle \tilde{x}''(\xi)-\tilde{y}''(\xi),\frac{\tilde{x}'(t)}{v_E}\right\rangle d\xi\right)d\tau \\
&\geqslant 0+(v_E-v_P)(s-t)-\frac{v_E\bar{u}+v_P\bar{w}}{2}(s-t)^2=\sigma(s-t),
\end{aligned}$$

for $\quad s\in[t,t+H]$.

LEMMA 2.4. *For any $h\in(0,\infty)$ there exists $h^*\in(0,h]$ and $d\in(0,\infty)$ such that, if $(t,a,b)\in[0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3$, $\|(a_1,a_2)-(b_1,b_2)\|\leqslant d$, $x\in X(t,a)$ and $y\in Y(t,b)$, then for $s^*=t+h^*$ we have $(s^*,x(s^*),y(s^*))\in M$.*

Proof. Let $h\in(0,\infty)$ be fixed and

$$h^*=\min\left\{1,h,\frac{v_E-v_P}{3\bar{u}(2v_E+v_P+v_E\bar{u})}\right\},$$

$$d=\frac{v_E-v_P}{3}h^*.$$

Take $(t,a,b)\in[0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3$ such that $\|(a_1,a_2)-(b_1,b_2)\|\leqslant d$, $x\in X(t,a)$, $y\in Y(t,b)$ and put $\tilde{x}=(x_1,x_2),\tilde{y}=(y_1,y_2)$. Then for $s\in[t,t+1]$ we obtain

$$\langle \tilde{x}'(s), \tilde{x}(s) - \tilde{y}(s)\rangle$$

$$= \left\langle \tilde{x}'(t) + \int_t^s \tilde{x}''(\xi)\, d\xi, \tilde{x}(t) - \tilde{y}(t) + (s-t)(\tilde{x}'(t) - \tilde{y}'(t)) \right.$$

$$\left. + \int_t^s \left(\int_t^\tau (\tilde{x}''(\xi) - \tilde{y}''(\xi))\, d\xi\right) d\tau \right\rangle$$

$$= \langle \tilde{x}'(t), \tilde{x}(t) - \tilde{y}(t)\rangle + (\|\tilde{x}'(t)\|^2 - \langle \tilde{x}'(t), \tilde{y}'(t)\rangle)(s-t)$$

$$+ \left\langle \tilde{x}'(t), \int_t^s \left(\int_t^\tau (\tilde{x}''(\xi) - \tilde{y}''(\xi))\, d\xi\right) d\tau \right\rangle$$

$$+ \left\langle \int_t^s \tilde{x}''(\xi)\, d\xi, \tilde{x}(t) - \tilde{y}(t) \right\rangle$$

$$+ \left\langle \int_t^s \tilde{x}''(\xi)\, d\xi, \tilde{x}'(t) - \tilde{y}'(t) \right\rangle (s-t)$$

$$+ \left\langle \int_t^s \tilde{x}''(\xi)\, d\xi, \int_t^s \left(\int_t^\tau (\tilde{x}''(\xi) - \tilde{y}''(\xi))\, d\xi\right) d\tau \right\rangle$$

$$\geqslant - v_E d + v_E(v_E - v_P)(s-t) - v_E^2 \bar{u}\,(s-t)^2 - v_E \bar{u} d(s-t)$$

$$- v_E \bar{u}(v_E + v_P)(s-t)^2 - v_E^2 \bar{u}^2\,(s-t)^3$$

$$\geqslant - v_E d + v_E(v_E - v_P - \bar{u}d)(s-t) - v_E \bar{u}\,(2v_E + v_P + v_E\bar{u})(s-t)^2.$$

Hence taking $s^* = t + h^*$ and observing that $(v_E - v_P)h^*/3 \leqslant (v_E - v_P)/3\bar{u}$ we obtain

$$\langle \tilde{x}'(s^*), \tilde{x}(s^*) - \tilde{y}(s^*)\rangle \geqslant -v_E d + v_E(v_E - v_P - \bar{u}d)\, h^*$$

$$- v_E \bar{u}\,(2v_E + v_P + v_E \bar{u})(h^*)^2$$

$$= - v_E d + [v_E - v_P - \bar{u}d - \bar{u}(2v_E + v_P + v_E \bar{u})\, h^*]\, v_E h^*$$

$$\geqslant - v_E d + \left[\frac{2}{3}(v_E - v_P) - \frac{1}{3}(v_E - v_P)\right] v_E h^* = 0.$$

Thus, it follows from the definition of the set M that $(s^*, x(s^*), y(s^*)) \in M$.

LEMMA 2.5. *There is $\delta_0 \in (0, \infty)$ and increasing function $\tilde{\sigma}$: $[0, \infty) \to [0, \infty)$ satisfying the inequality $\tilde{\sigma}(r) \leqslant r$, for $r \in [0, \infty)$ and such that for any*

$(t, a, b) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$ one can find $x^0 \in X(t, a)$ which satisfies the inequality

$$\|(x_1^0(s), x_2^0(s)) - (y_1(s), y_2(s))\| \geqslant \tilde{\sigma}(\|(a_1, a_2) - (b_1, b_2)\|),$$

for all $y \in Y(t, b)$ and $s \in [t, t+\delta_0]$.

Proof. Following to the suitable considerations from [1] one can prove that there is $\delta_0 \in (0, \infty)$ and increasing function $\sigma^*\colon [0, \delta_0] \to [0, \infty)$ such that for any $(t, a, b) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$ there exists $x^0 \in X(t, a)$ satisfying the inequality

$$\|(x_1^0(s), x_2^0(s)) - (y_1(s), y_2(s))\| \geqslant \sigma^*(s-t),$$

for all $y \in Y(t, b)$ and $s \in [t, t+\delta_0]$. It follows from Lemma 2.1 that for any $x \in X(t, a)$, $y \in Y(t, b)$ and $s \in [t, \infty)$ the following inequality

$$\begin{aligned} &\|(x_1(s), x_2(s)) - (y_1(s), y_2(s))\| \\ &\qquad \geqslant \|(a_1, a_2) - (b_1, b_2)\| - \left(\sqrt{v_E^2 + \bar{u}^2} + \sqrt{v_P^2 + \bar{w}^2}\right)(s-t) \end{aligned}$$

is satisfied. Therefore for $s \in [t, t+\delta_0]$ we have

$$\begin{aligned} &\|(x_1^0(s), x_2^0(s)) - (y_1(s), y_2(s))\| \\ &\qquad \geqslant \min\{\max\{\sigma^*(s-t), \|(a_1, a_2) - (b_1, b_2)\| \\ &\qquad - \left(\sqrt{v_E^2 + \bar{u}^2} + \sqrt{v_P^2 + \bar{w}^2}\right)(s-t)\} : s \in [t, t+\delta_0]\}. \end{aligned}$$

Put

$$\tilde{\sigma}(r) = \min\{\max\{\sigma^*(\tau), r - \left(\sqrt{v_E^2 + \bar{u}^2} + \sqrt{v_P^2 + \bar{w}^2}\right)\tau\} : \tau \in [0, \delta_0]\},$$

for $r \in [0, \infty)$. The proof of Lemma 2.5 is complete.

Now we are going to prove Theorem 2.1. Let us fix $\varepsilon, T \in (0, \infty)$ and put

$$\tilde{n} = \min\{n \in \mathbb{N} : T \leqslant nH\}$$

(where H is taken from Lemma 2.3) and

$$\delta_1 = T/\tilde{n}.$$

It follows from Lemma 2.2 that there exists a sequence $\{\varepsilon_0, \varepsilon_1, ..., \varepsilon_{\tilde{n}}\}$ which satisfies the following conditions:

$$0 < 3\varepsilon_i \leqslant \varepsilon_{i+1}, \qquad \text{for} \quad i = 0, 1, ..., n-1,$$

$\varepsilon_{\tilde{n}} = \varepsilon/2$, for arbitrary $t \in [0, \infty)$, a, $\tilde{a} \in \mathbb{R}^3$ and for $i \in \{0, 1, \dots, \tilde{n} - 1\}$, if $\|a - \tilde{a}\| \leqslant 3\varepsilon_i$, then $\|x[t, a, u](s) - x[t, \tilde{a}, u](s)\| \leqslant \varepsilon_{i+1}$, for all $u \in U$ and $s \in [t, t + T]$. Take

$$h_1 = \frac{\varepsilon_0}{3\sqrt{v_E^2 + \bar{u}^2}},$$
$$h = \min\{h_1, \delta_1, \delta_0\}$$

(where δ_0 is taken from Lemma 2.5)

$$h_2 = \min\left\{\frac{d}{3\sqrt{v_E^2 + \bar{u}^2}}, \frac{d}{3\sqrt{v_P^2 + \bar{w}^2}}\right\}$$

(where d is chosen for h conformably to Lemma 2.4)

$$\tilde{m}_1 = \min\{m \in \mathbb{N}: \delta_1 \leqslant mh_2\}$$

and, finally,

$$\delta_2 = \delta_1/\tilde{m}_1.$$

It is easy to see that $T/\delta_2 = (T/\delta_1)(\delta_1/\delta_2) = \tilde{n}\tilde{m}_1 \in \mathbb{N}$. Thus, take

$$\tilde{m} = \tilde{n}\tilde{m}_1.$$

It follows from Lemma 2.2 that there exists a sequence $\{\eta_0, \eta_1, \dots, \eta_{\tilde{m}-1}\}$ satisfying the following conditions:

$$0 < 2\eta_j \leqslant \eta_{j+1}, \qquad j = 0, 1, \dots, \tilde{m} - 2,$$
$$\eta_{\tilde{m}-1} = \varepsilon_0,$$

for arbitrary $t \in [0, \infty)$, $a, \tilde{a} \in \mathbb{R}^3$ and $j \in \{0, 1, \dots, \tilde{m} - 2\}$, if $\|a - \tilde{a}\| \leqslant 2\eta_j$, then $\|x[t, a, u](s) - x[t, \tilde{a}, u](s)\| \leqslant \eta_{j+1}$, for all $u \in U$ and $s \in [t, t + \delta_2]$.

Finally, for $r \in [0, \infty)$ let

$$\rho_1(r) = \tfrac{1}{2}\min\{\max\{\sigma(\tau), \tilde{\sigma}(r) - (v_E + v_P)\,\tau\}: \tau \in [0, \delta_1]\},$$
$$\rho_2(r) = \min\{2\eta_0, \rho_1(r)\}$$

and

$$\rho(t, a, b) = \rho_2(r),$$

for all $(t, a, b) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$ satisfying the equality $\|(a_1, a_2) - (b_1, b_2)\| = r$.

Now, let us fix $u \in U$. We will define the functions f and δ.

If $(t, a, b) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$ and $\|(a_1, a_2) - (b_1, b_2)\| > d$ or $(t, a, b) \in M$ (where M is defined before Lemma 2.3), then we put

$$f(t, a, b) = x[t, a, u].$$

If $\|(a_1, a_2) - (b_1, b_2)\| \leqslant d$ and $(t, a, b) \notin M$, take

$$(f(t, a, b))(s) = x^0(s), \qquad \text{for } s \in [t, t+h^*],$$

and

$$(f(t, a, b))(s) = x[t+h^*, x^0(t+h^*), u](s), \qquad \text{for } s \in (t+h^*, \infty),$$

where x^0 is chosen for (t, a, b) conformably to Lemma 2.5 and h^* is chosen for h conformably to Lemma 2.4.

The function δ we define in the following way. If $\|(a_1, a_2) - (b_1, b_2)\| > d$, then we put

$$\delta(t, a, b) = \delta_2.$$

If, conversely, $\|(a_1, a_2) - (b_1, b_2)\| \leqslant d$, take

$$\delta(t, a, b) = \delta_1.$$

Let us fix $(t, a, b) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$ and $y \in Y(t, b)$. Suppose that $[g, \delta, a, b, t]$ is ρ-closed to $[f, \delta, a, b, t]$ and $x = [g, \delta, a, b, t](y)$. At first we will prove that $(x_1(s), x_2(s)) \neq (y_1(s), y_2(s))$ for $s \in [t, t+T]$. Assume that $\|(a_1, a_2) - (b_1, b_2)\| > d$. Then $\delta(t, a, b) = \delta_2$ and $f(t, a, b) = x[t, a, u]$. From the definitions of h_2 and δ_2 it follows that $\delta_2 \leqslant h_2$, so $\sqrt{v_E^2 + \bar{u}^2}\, \delta_2 \leqslant d/3$ and $\sqrt{v_P^2 + \bar{w}^2}\, \delta_2 \leqslant d/3$. By Lemma 2.1, for all $\tilde{x} \in X(t, a)$ and $s \in [t, t+\delta_2]$ we have

$$\begin{aligned}\|(\tilde{x}_1(s), \tilde{x}_2(s)) - (y_1(s), y_2(s))\| &\geqslant \|(a_1, a_2) - (b_1, b_2)\| \\ &\quad - \|(\tilde{x}_1(s), \tilde{x}_2(s)) - (a_1, a_2)\| - \|(y_1(s), y_2(s)) - (b_1, b_2)\| \\ &\geqslant d - (\sqrt{v_E^2 + \bar{u}^2} + \sqrt{v_P^2 + \bar{w}^2})(s-t) \geqslant d/3.\end{aligned}$$

In the case $\|(a_1, a_2) - (b_1, b_2)\| \leqslant d$ and $(t, a, b) \notin M$ we have

$$\delta(t, a, b) = \delta_1,$$

$$(f(t, a, b))(s) = x^0(s), \qquad \text{for } s \in [t, t+h^*],$$

and

$$(f(t, a, b))(s) = x[t+h^*, x^0(t+h^*), u](s), \qquad \text{for } s \in (t+h^*, \infty).$$

By Lemma 2.5,

$$\begin{aligned}
&\|((f(t,a,b))_1(s),(f(t,a,b))_2(s))-(y_1(s),y_2(s))\| \\
&\quad\geqslant \tilde{\sigma}(\|(a_1,a_2)-(b_1,b_2)\|),
\end{aligned} \tag{2.1}$$

for $s \in [t, t+h^*]$ and, according to Lemma 2.4,

$$\begin{aligned}
&\|((f(t,a,b))_1(s),(f(t,a,b))_2(s))-(y_1(s),y_2(s))\| \\
&\quad\geqslant \sigma(s-(t+h^*)),
\end{aligned} \tag{2.2}$$

for $s \in [t+h^*, t+h^*+H]$. By the inequality (2.1),

$$\begin{aligned}
&\|((f(t,a,b))_1(s),(f(t,a,b))_2(s))-(y_1(s),y_2(s)\| \\
&\quad\geqslant \|((f(t,a,b))_1(t+h^*),(f(t,a,b))_2(t+h^*)) \\
&\qquad -(y_1(t+h^*),y_2(t+h^*))\| \\
&\qquad -\|((f(t,a,b))_1(s),(f(t,a,b))_2(s)) \\
&\qquad -((f(t,a,b))_1(t+h^*),(f(t,a,b))_2(t+h^*))\| \\
&\qquad -\|(y_1(s),y_2(s))-(y_1(t+h^*),y_2(t+h^*))\| \\
&\quad\geqslant \tilde{\sigma}(\|(a_1,a_2)-(b_1,b_2)\|)-(v_E+v_P)(s-(t+h^*)),
\end{aligned}$$

for $s \in [t+h^*, \infty)$. Therefore, by (2.2), if $s^* = t+h^*$, then

$$\begin{aligned}
&\|(x_1(s),x_2(s))-(y_1(s),y_2(s))\| \\
&\quad\geqslant \|(y_1(s),y_2(s))-((f(t,a,b))_1(s),(f(t,a,b))_2(s))\| \\
&\qquad -\|(x_1(s),x_2(s))-((f(t,a,b))_1(s),(f(t,a,b))_2(s))\| \\
&\quad\geqslant \min\{\max\{\sigma(s-s^*),\tilde{\sigma}(\|(a_1,a_2)-(b_1,b_2)\|) \\
&\qquad -(v_E+v_P)(s-s^*)\}: s\in[s^*,s^*+H]\}-\rho(t,a,b),
\end{aligned}$$

for $s \in [s^*, s^*+H]$. It is easy to see that for $s \in [t, s^*]$ the same inequality holds. Thus from the definition of ρ and the inequality $\delta_1 \leqslant H$ it follows that $(x_1(s), x_2(s)) \neq (y_1(s), y_2(s))$, for $s \in [t, t+\delta_1]$.

In the case $\|(a_1, a_2)-(b_1, b_2)\| \leqslant d$ and $(t, a, b) \in M$ the proof of the above condition is similar.

Now we give the sketch of the proof of the inequality

$$\|x(s)-x[t,a,u](s)\| \leqslant \varepsilon, \qquad \text{for} \quad s \in [t, t+T].$$

Let C be the set, described in Proposition 1.3, determined by $[g, \delta, a, b, t]$ and y and let $\{t_0, t_1, \ldots, t_k\}$ be such that

$$t_0 = t, \qquad C \cap [t, t+T] = \{t_0, t_1, \ldots, t_{k-1}\}, \qquad t_k = \min\{s \in C: t_{k-1} < s\}.$$

Of course, $t_k > t+T$. It follows from the formula $\delta([0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3) = \{\delta_1, \delta_2\}$ that the inequalites $1 \leqslant k \leqslant \tilde{n} \leqslant \tilde{m}$ hold.

Assume that $\|(a_1, a_2) - (b_1, b_2)\| \leqslant d$. (The proof in the case $\|(a_1, a_2) - (b_1, b_2)\| > d$ is similar.) Denote by $\{k_0, k_1, \ldots, k_j\}$ the sequence of natural numbers with the following properties:

$$\begin{aligned}
&0 = k_0 < k_1 < \cdots < k_j = k, \\
&\|(x_1(t_l), x_2(t_l)) - (y_1(t_l), y_2(t_l))\| \leqslant d, \\
&\text{iff} \quad l \in \{k_0, k_1, \ldots, k_{j-1}\}, \qquad \text{for} \quad l = 0, 1, \ldots, k-1.
\end{aligned}$$

Further, notice that $\|x(t_0) - x[t, a, u](t_0)\| = 0 < 2\varepsilon_0$. Now, taking account of the properties of the sequences $\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{\tilde{n}}\}$, $\{\eta_0, \eta_1, \ldots, \eta_{\tilde{m}-1}\}$, the inequality $\rho(r) \leqslant 2\eta_0$, for $r \in [0, \infty)$ and the inequalites $2\eta_0 \leqslant \eta_l \leqslant 2\varepsilon_0 < 3\varepsilon_i$, for $l = 0, 1, \ldots, \tilde{m}-1$, $i = 0, 1, \ldots, j$, one can prove, by induction, that $\|x(s) - x[t, a, u](s)\| \leqslant 2\varepsilon_i$, for $s \in [t_0, t_{k_i}]$ and $i = 0, 1, \ldots, j$. Thus observing that $2\varepsilon_i \leqslant 2\varepsilon_{\tilde{n}} = \varepsilon$ and using the inequality $j \leqslant \tilde{n}$ we end the proof of Theorem 2.1.

Corollary 2.1. *If $v_E > v_P$ and $v_F \bar{u} \geqslant v_P \bar{w}$, then for all $n \in \mathbb{N}$ the player E wins along each trajectory in the game $(X, Y)_{1+n}$.*

Proof. This Theorem immediately follows from Theorem 2.1 and Corollary 1.1.

Corollary 2.2. *Let $Z \subset \mathbb{R}^3$ and $(t, a, b) \in \mathbb{R}^3$ be defined as in Example 1.1. Then for all $n \in \mathbb{N}$ and $b^i \in \mathbb{R}^3$, $b^1 = b$, $(a_1, a_2) \neq (b_1^i, b_2^i)$, for $i = 1, 2, \ldots, n$, there exists $e \in E(X, Y, a, b^1, \ldots, b^n, t)$ which keeps the player E in the set Z and wins in the game $(X, Y, a, b^1, \ldots, b^n, t)$.*

Proof. This is a consequence of Proposition 1.4 and Corollary 2.1.

Remark 2.1 (See [1]). If $v_E < v_P$ or $v_F = v_P$ and $\bar{u} \leqslant \bar{w}$, then there exist a, $b \in \mathbb{R}^3$ such that $(a_1, a_2) \neq (b_1, b_2)$ and the player P wins in the game (X, Y, a, b, t), for $t \in [0, \infty)$.

Remark 2.2. The known sufficient conditions for the existence of the evasion strategy, see, e.g., [2, 4], cannot be applied to the described game (even in the case $n = 1$).

Remark 2.3. Using the more precise calculations one can prove that, if

$v_E > v_P$ and $v_E \bar{u} \geqslant v_P \bar{w}$, then for any $t \in [0, \infty)$ and $a, b^i \in \mathbb{R}^3$ such that $(a_1, a_2) \neq (b_1^i, b_2^i)$, $i = 1, 2, \ldots, n$, there exists $e \in E(X, Y, a, b^1, \ldots, b^n, t)$ satisfying the inequality

$$\inf\{\|((e(y^i))_1(s), (e(y^i))_2(s)) - (y_1^i(s), y_2^i(s))\| : s \in [t, \infty)\} > 0,$$

for all $y = (y^1, \ldots, y^n) \in Y(t, b^1) \times \cdots \times Y(t, b^n)$.

Remark 2.4. Using the above methods one can prove that for each $n \in \mathbb{N}$ the player E wins along each trajectory in the game $(X, Y)_{1+n}$, where Y is defined as above but X is determined by the differential equation

$$x_1'(s) = u_1(s),$$
$$x_2'(s) = u_2(s), \qquad u_1^2(s) + u_2^2(s) \leqslant v_E^2.$$

(The case called "The Homicidal Chauffeur Game" [3]).

To this end it is enough to prove five lemmas analogous to Lemmas 2.1–2.5. It is not hard to formulate and to prove these lemmas. (In order to prove an equivalent of Lemma 2.3 it is convenient to assume, without loss of the generality, that $v_E < v_P$).

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